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OPTIMAL CONTROL OF A GRADED MANPOWER SYSTEM

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and
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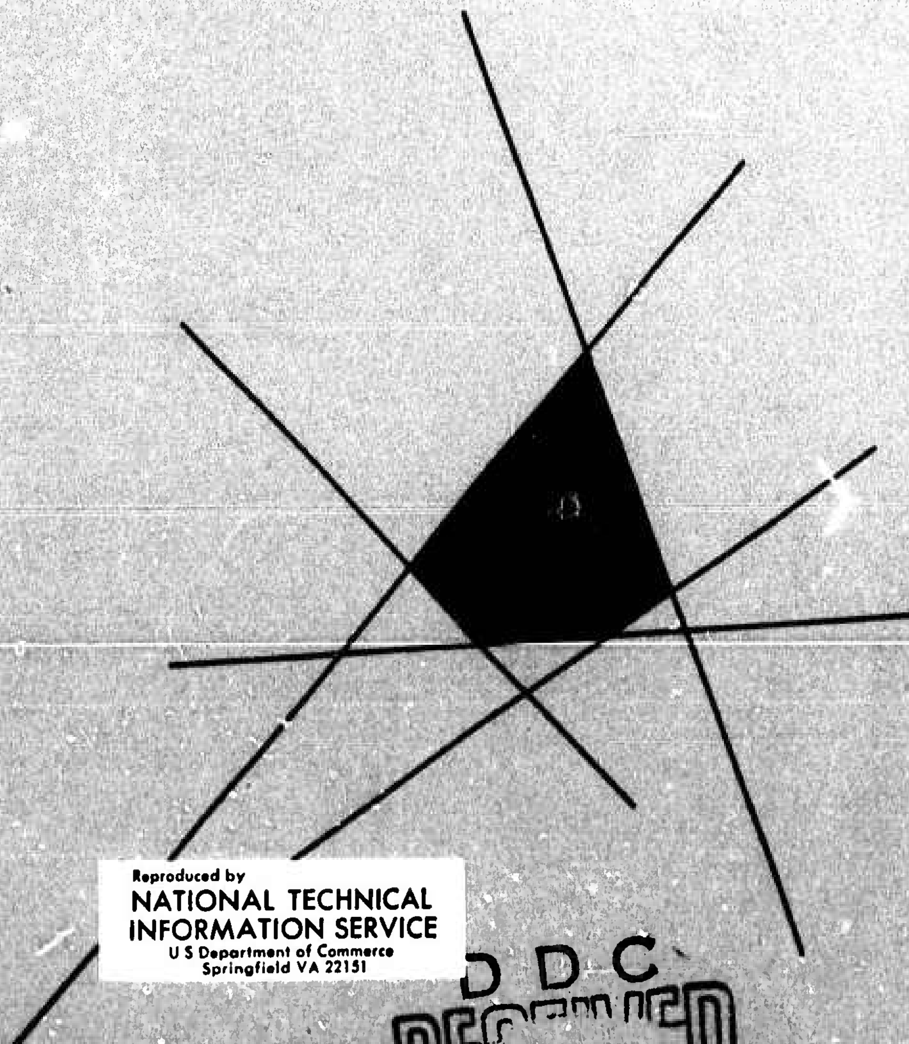

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OPTIMAL CONTROL OF A GRADED MANPOWER SYSTEM

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ABSTRACT

We consider a fractional flow model of a graded manpower system and develop algorithms for calculating optimal control policies in four situations: (i) finite time horizons with no constraints on staff distributions, (ii) finite time horizon with constraints on final staff distribution, (iii) infinite horizon with constraints on staff distribution and (iv) problems with a nonstationary transient stage and an infinite stationary stage. In each case results developed in solving the simpler problems are useful in analyzing more complicated situations.

In addition to providing computational procedures we apply the algorithms to a three rank model and discuss the possible uses and limitations of our procedure.

INTRODUCTION

This paper studies a fractional flow model of a graded manpower system. A useful mathematical programming procedure for evaluation of alternate policies is presented, and some illustrative examples given.

The model is designed as an aggregate planning device; it is not designed to specify hiring and promotion policy in an exact sense, but to assist in answering questions of the following sort: (i) How should the growth of a new organization be scheduled? (ii) What are the costs of institutional restrictions on staff? (iii) How would the system react to a change in promotion policy? (iv) What is the relation between the cost of operation and the rate of growth? (v) What impact would a wage increase have on the cost of future operations?

An extended discussion of similar models in manpower planning can be found in Bartholomew [1]. This paper follows Bartholomew [2] in the basic model structure and search for optimal policies.

Although models of this type are often called Markov models, we believe this name is misleading. The Markov interpretation implies there is a stochastic decision rule that governs promotion policy for each individual in the system. We must either assume that all individuals are the same and face the same stochastic promotion mechanism or that different classes of individuals face different promotion possibilities and that the given promotion matrix can describe the aggregate behavior of all classes. Both of these assumptions are difficult to defend. In addition, if the coefficients P_{ij} truly represent probabilities, then the variances of random variables such as the number of years in the system, and the total salary received should be meaningful. Our rough calculations have indicated that these variances are far too large to be consistent with observed

behavior. Therefore we prefer the deterministic fractional flow interpretation of the model. The organization *as a matter of policy* decides to promote a fraction of people in rank 1 to rank 2 each year. Our model is designed to explore the consequences of that policy decision.

The system has n ranks $1, 2, \ldots, n$. In period t the n vector $x(t) = [x_1(t), \ldots, x_n(t)]$ describes the distribution of manpower among the ranks. We assume the initial distribution $x(0)$ is known. An $n \times n$ promotion matrix P governs transitions within the system, where P_{ij} is the fraction of workers in rank i who move to rank j. We generally assume that P is independent of t. This assumption is crucial for the infinite horizon cases in Section III and part of IV. In Sections I and II, however, P can vary with t.

The system hiring policy is described by the n-vector $u(t) = [u_1(t), \ldots, u_n(t)]$ where $u_j(t)$ is the number of new appointments in rank j made during period t. We shall assume throughout that $u(t)$ is nonnegative: there is no firing. We adopt the convention that these new workers start work in period $t + 1$, therefore

$$x_j(t + 1) = \sum_{i=1}^{n} x_i(t)P_{ij} + u_j(t) .$$

In matrix notation

$$x(t + 1) = x(t)P + u(t) . \tag{1}$$

Equation (1) is the system's Law of Motion.

We assume the system grows at rate $(\theta - 1)$. We have expansion, no growth, or contraction as $\theta > 1$, $\theta = 1$, or $\theta < 1$. Let e be an n dimension column vector with each component equal to one. At time t the growth constraint is

$$x(t)e = \theta^t x(0)e \quad \text{for} \quad t \geq 1 .$$

Alternate growth constraints are possible. For example, let f be an n dimension column vector with f_i equal to the total annual institutional financial support given to a worker in rank i. Then $x(t)f$ gives the total manpower budget in period t. The growth constraint could also read

$$x(t)f = \theta^t x(0)f \quad \text{for} \quad t \geq 1 \tag{2}$$

to reflect a changing manpower budget. A strict cost interpretation for (2) is not necessary: for example f_i could represent some measure of the productivity of rank i staff members. In that case, condition (2) would give the total system productivity requirement for the t^{th} period.

In the optimization models described below our objective is a linear function of the state vector $x(t)$ of staff distribution and the control vector $u(t)$ of new appointments. Throughout we interpret this objective as the cost of system operations. There are two reasons for this: first it eases the exposition, second it is a useful criterion. The minimum cost of system operations is a lower bound on the cost that could be incurred using any other policy. It is reasonable to assume that the institution's personnel policy will depart from the minimum cost policy indicated by our model. However, knowledge of the minimum cost policy also provides a direction of policy change, if the institution desires to change policies and reduce costs. As with the growth constraint (2), alternative interpretations of the linear objective are allowable. We could, for example, use the objective function to measure productivity, and seek to *maximize* it subject to budgetary or size restrictions. In essence, all the results in this paper would carry over in this case.

In Section I we introduce a finite horizon planning problem with no constraints on the staff distribution and develop an efficient dynamic programming procedure to solve this problem. Section II considers a slight generalization of the finite

horizon planning problem: constraints are placed on the final distribution $x(T)$. This problem is solved using generalized linear programming, where the dynamic programming algorithm described in Section I is the column generating subproblem. Section III considers infinite horizon problems and computes lower bounds on the optimal value. We obtain asymptotically optimal stationary linear decision rules. This easily implemented policy is optimal if the resulting sequence of staff distributions $x(t)$ does not violate the system constraint. In Section IV the theory of Sections II and III are blended together to examine an infinite horizon planning problem that combines a T year transient stage and an indefinite stationary stage. As in Section III we are able to derive lower bounds on the optimal value of this two stage planning problem.

We solved the problems presented in Sections I-IV using a dynamic programming and generalized linear programming. All of these problems could have been attacked directly by solving larger linear programs. We took the dynamic programming generalized LP approach for several reasons. Our analysis of the problem developed in exactly the way our results are presented. We first developed the DP procedure to study the free end point problem, and then found that these results could be used to help solve the more complicated problems with terminal constraints. We feel that the generalized LP approach is a more natural way to analyze the problems presented in Sections II and IV since it focuses attention on properties of sets in the n dimension space of staff distributions. In addition, the dynamic programming procedure is efficient and application of generalized LP is new and of independent interest. Our computations were done interactively, and we frequently changed the data and policy environment to examine the effects of these changes on appointment policies and staff distributions. The compact form of the data storage needed in the generalized LP approach saved us from regenerating large arrays each time the problem was changed.

We did not test each method to see which was numerically more efficient on a set of test problems. That type of analysis seemed far removed from the purpose of the paper. We were, however, satisfied with the speed and storage requirements of the approach we selected.

In Section V we examine a three state university planning model and give two examples of how the theory developed in Sections I-IV can be used. The example's small size is consistent with the purpose of the model. The organization's policy is as much determined by the growth factor θ, and the promotion matrix P as the hiring policy $u(t)$. A small model allows us to do extensive sensitivity analysis to determine the effects of changing θ, P, and other system parameters. A reader interested in the model's value as an aid to decisionmakers can turn directly to Section V without reading further.

The following definitions and notations are used. The symbol $\triangleq$ means "is defined to be". Given the vector f with $f_i > 0$ we say

$$S_t \triangleq \{x(t) \mid x(t)f = \theta^t xf, x(t) \geq 0\}$$

where x is the initial distribution. S_t is the set of possible distribution at time t, assuming a growth rate of $(\theta - 1)$ in each period, and $S_0 \triangleq S$.

Let $R(x) \triangleq \{y \mid y \geqq xP, yf = \theta xf\}$. Starting at x, $R(x)$ is the set of points that can be reached in one period. For any t and set $A \subseteqq S_t$

$$R(A) \triangleq \bigcup_{x \in A} R(x) = \{y \mid y \in R(x), x \in A\} .$$

$R(A)$ is the set of points that can be reached in one period from the set A. Let $R^1(A) \triangleq R(A)$, and for $t \geqq 2$,

$$R^t(A) = R(R^{t-1}(A)) .$$

$R^t(A)$ is the set of points that can be reached from A in exactly t periods.

Let

$$E \triangleq \{y \mid \theta y \geqq yP, y \geqq 0\} .$$

If $y \in E$, then $x(t) = \theta^t y$ is a sequence that obeys the size constraint (2) and the law of motion (1) with $u(t) = \theta y - yP$. Note that $E = \{y \mid \theta y \in R(y), y \geq 0$

In Section III we designate the promotion matrix as Q and the growth factor as δ. This allows us to distinguish the transient and stationary promotion matrices and growth rates when they are united in Section IV. We denote the empty set as ϕ, and use cl for closure and ri for relative interior.

I. MINIMUM COST OPERATION: FINITE TIME HORIZON - NO TERMINAL CONSTRAINTS.

This section develops an efficient method for minimizing the cost of operations over a T year horizon with no terminal constraints.

The distribution $x(t)$ follows the law of motion described in the introduction with the quantity $x(t)f$ growing at a constant rate $\theta - 1$. The present value of cost incurred in period t is $x(t)c(t) + u(t)d(t)$. There is a reward $x(T)q$ (perhaps zero) attached to the terminal distribution $x(T)$. The present value of all costs is thus

$$\sum_{t=0}^{T-1} x(t)c(t) + u(t)d(t) - x(T)q \ .$$

We assume the current distribution $x(0) = x$ is known.

The problem is

$$\text{(1)} \qquad \text{Minimize} \sum_{t=0}^{T-1} x(t)c(t) + u(t)d(t) - x(T)q$$

Subject to

$$\begin{aligned} x(t+1) &= x(t)P + u(t) \\ x(t)f &= \theta^t xf \\ u(t) &\geq 0 \qquad t = 0,1,2, \ldots, T-1 \\ x(0) &= x \geq 0 \end{aligned}$$

The first set of constraints is the law of motion that describes distribution changes over time. The second set of constraints indicates that the quantity $x(t)f$ grows at a constant rate $(\theta - 1)$. The third set of constraints says that no firing takes place.

Note that

$$x(t + 1)f = x(t)Pf + u(t)f = \theta x(t)f$$

Thus $u(t)f = x(t)[\theta I - P]f$. Let $v = [\theta I - P]f$. Our solution procedure below *assumes* v *is strictly positive* and $P \geq 0$. When $f = e$, constraint $u(t)e = x(t)v = x(t)e(\theta - 1) + x(t)(I - P)e$, says the number of people hired is equal to the number that leave, $x(t)(I - P)e$, plus the number needed to increase size, $(\theta - 1)x(t)e$.

Although we shall not work out the details, the analysis below is still valid if P is a function $P(t)$ of time and $v(t) = [\theta I - P(t)]f$ is strictly positive for all t .

Problem (1) can be solved with an efficient dynamic programming technique. For $s = 0,1,2 \ldots T$ define $V(x,s)$ to be the optimal value of

$$\text{(2)} \qquad \text{Minimize} \quad \sum_{t=s}^{T-1} x(t)c(t) + u(t)d(t) - x(T)q$$

$$\text{Subject to} \quad x(t + 1) = x(t)P + u(t)$$

$$u(t)f = x(t)v$$

$$u(t) \geq 0 \qquad t = s,s+1, \ldots, T-1$$

$$x(s) = x \geq 0 \qquad \text{given}$$

Note that $V(x,0)$ is the optimal value of (1) as a function of the initial conditions $x(0) = x$ and that $V(x,T) = -xq$. The optimal value function obeys the usual dynamic programming principle of optimality.

$$\text{(3)} \qquad V(x,t) = xc(t) + \min_{\substack{uf=xv \\ u \geq 0}} \left[ud(t) + V(xP + u,t + 1)\right]$$

Assume $\qquad V(x,t + 1) = xh(t + 1)$

a linear function.

For $t = T - 1, h(T) = -q$. Thus this assumption is valid for $t = T - 1$. Then equation (3) becomes

(4) $$V(x,t) = x[c(t) + Ph(t+1)] + \min_{\substack{uf = xv \\ u \geq 0}} [u(d(t) + h(t+1))]$$

Since $v > 0$, xv is strictly positive for any semi-positive x. Thus the linear program in this functional equation has a trivial solution. Let $\pi(t)$ be the index such that

(5) $$\eta = \frac{[d(t) + h(t+1)]_{\pi(t)}}{f_{\pi(t)}} = \min_{i=1,2,\ldots,N} \frac{[d(t) + h(t+1)]_i}{f_i}$$

The optimal solution to the linear program in (4) is

$$u_i = \left\{ \begin{matrix} 0 & \text{if} & i \neq \pi(t) \\ \frac{xv}{f_i} & \text{if} & i = \pi(t) \end{matrix} \right\}$$

Therefore

$$V(x,t) = x\{c(t) + Ph(t+1)\} + xv\eta \; .$$

Set

(6) $$h(t) = c(t) + Ph(t+1) + \eta v \; .$$

By this constructive induction we have demonstrated

<u>Theorem 1</u>:

The optimal solution of problem (1) is to set

$$u_i(t) = \left\{ \begin{matrix} \frac{xv}{f_i} & \text{if} \quad i = \pi(t) \\ 0 & \text{otherwise} \end{matrix} \right\} .$$

The index $\pi(t)$ and the vector $h(t)$ are calculated recursively from (5), (6), and the initial condition $h(T) = -q$.

A more general analysis of this no terminal constraint model can be found in Section 1 of [7]. The case of nondecreasing size is treated and a particular class of problems is shown to have a closed form solution. However, we wish to use this model as a building block in solving more detailed problems with terminal constraints.

It is possible, by eliminating the variables $x(t)$ from problem (1), to express (1) as a linear program with T equality constraints and $n \times T$ variables. In choosing the dynamic programming approach to the solution of problem (1), we considered several possible advantages and disadvantages of the two methods. First, it must be understood that we were doing interactive computations. The interactive approach is most useful if the programs allow us to quickly analyze the effects of alternate policies or data inputs. In other words we could change P , f , c , d , $x(0)$, and q easily and resolve the problem. The dynamic program offered greater flexibility in changing the data and it also had a much smaller storage requirement. Both of these qualities are important in interactive computations. The sole disadvantage we saw with the dynamic programming approach was the necessity to code a special algorithm. In contrast, interactive LP codes are already available. However, to analyze a model interactively using linear programming it is necessary to write a matrix generation program that constructs the LP constraint matrix from the problem's data and then retranslates the LP solution into the terms of the original problem. We found it was as easy to write the dynamic programming code as to construct the matrix generator for the LP. In addition, it seemed unlikely to us that an LP could improve on the extremely simple calculations required in (5) and (6). Finally, we shall in later sections be faced with similar choices in solution strategies for more complicated problems. In those situations we believed that the dynamic programming

approach offered even greater advantages. Thus it was necessary to write the dynamic programming code for use as a subroutine in the more complicated problems.

II. MINIMAL COST OPERATION: A FINITE HORIZON AND A TERMINAL CONSTRAINT.

This section refines the model presented in Section I by adding the terminal constraint, $x(T) \in C = \{y \mid yA \geq 0\}$. Since $S_T = \{y \mid y = \theta^{1} xf, y \geq 0\}$, the terminal constraint can also be written as

$$x(T) \in C \cap S_T \triangleq B \, .$$

Any polyhedral subset B of S_T can be represented in the manner described above.

As one example consider a restriction that the fraction of workers in ranks $h + 1, h + 2, \ldots, N$ is less than δ. This can be written

$$\delta \sum_{i=1}^{h} x_i(T) + (\delta - 1) \sum_{i=h+1}^{N} x_i(T) \geq 0 \, .$$

We can also specify final target distribution $x(T)$.

If there are K terminal constraints then A is an $N \times K$ matrix. The optimization problem is

(1) $$\text{Minimize} \quad \sum_{t=0}^{T-1} x(t)c(t) + u(t)d(t)$$

$$\begin{aligned} \text{Subject to} \quad & x(t + 1) = x(t)P + u(t) \\ & u(t)f = x(t)v \\ & u(t) \geq 0 \quad t = 0, 1, \ldots, T-1 \\ & x(T)A \geq 0 \\ & x(0) = x \geq 0 \quad \text{given.} \end{aligned}$$

This is exactly problem (1) of Section I with $q = 0$, and an additional constraint $x(T)A \geq 0$.

Recall from the introduction that $R^T(x)$ is the set of all points that can be reached from x in exactly T steps. If $R^T(x) \cap C = \phi$, then problem (1) is infeasible: there is no $x(T)$ satisfying the terminal constraint.

For each $y \in R^T(x)$ define $W(y)$ as the optimal value of

$$\text{Minimum} \quad \sum_{t=0}^{T-1} x(t)c(t) + u(t)d(t)$$

$$\text{Subject to} \quad x(t+1) = x(t)P + u(t)$$

$$u(t)f = x(t)v$$

$$u(t) \geq 0$$

$$x(T) = y$$

$$x(0) = x \geq 0$$

$W(y)$ is the minimum cost of going from x to y in T steps. For $y \notin R^T(x)$ we define $W(y) = +\infty$. Problem (1) is equivalent to

$$\text{(2)} \qquad \text{Minimize} \quad W(y)$$

$$\text{Subject to} \quad y \in C \cap R^T(x)$$

This section shows how we can exploit the efficient dynamic programming procedure developed in Section I to solve the problem with a terminal constraint. The method is outlined below using Dantzig and Wolfe's generalized linear programming [4]. We maintain a feasible solution of problem (2) at all times and calculate a sequence of lower bounds on the optimal value of (2).

After describing the algorithm we will show how an optimal solution of (1)

is constructed from an optimal solution of (2). Then we give a theoretical justification of the algorithm and we present a phase I procedure that will either initialize the algorithm or show that problem (2) is infeasible.

Let $y^1, y^2, \ldots, y^m$ be points in $R^T(x)$, and let $W_i = W(y^i)$.

The *master* linear program is

$$
\text{(3)} \qquad \begin{aligned}
&\text{Minimize} && \sum_{i=1}^{m} \lambda_i W_i \\
&\text{Subject to} && \sum_{i=1}^{m} \lambda_i (y^i A) \geqq 0 \\
& && \sum_{i=1}^{m} \lambda_i = 1 \\
& && \lambda \geqq 0
\end{aligned}
$$

Let λ^m be the optimal solution of (3). Then $\sum_{i=1}^{m} \lambda_i^m y_i$ is *feasible* for (2) and $\sum_{i=1}^{m} \lambda_i^m W_i$ is an *upper bound* on the optimal value of (2).

Let (r^m, σ^m) be the optimal dual variables associated with problem (3). Note that $\sigma^m = \sum_{i=1}^{m} \lambda_i^m W_i$. The *subproblem* is

$$
\text{(4)} \qquad \begin{aligned}
&\text{Minimize} \quad W(y) - yAr^m \\
&\qquad y \in R^T(x) .
\end{aligned}
$$

Subproblem (4) is solved by setting $q^m = Ar^m$ and solving problem (1) of Section I. Let y^{m+1} be the optimal solution of (4) and let ν^{m+1} be the optimal value of (4). Then ν^{m+1} is a *lower bound* on the optimal value of (2) and

$$W_{m+1} = \nu^{m+1} + y^{m+1}Ar^m .$$

We consider two cases:

1. If $\nu^{m+1} < \sigma^m$, then column $(W_{m+1}, y^{m+1}A)$ is added to problem (3).

2. If $\nu^{m+1} \geqq \sigma^m$, then $\sum_{i=1}^{m} \lambda_i^m y_i$ is optimal for (2) and the optimal value of (2) is

$$\nu^{m+1} = \sigma^m = \sum_{i=1}^{m} \lambda_i^m W_i = W\left(\sum_{i=1}^{m} \lambda_i^m y^i\right) .$$

The generalized linear programming algorithm solves (3) and (4) while successively generating a sequence of upper and lower bounds on the optimal value. If optimal termination occurs on iteration m, then $\bar{y} = \sum_{i=1}^{m} \lambda_i^m y^i$ is the optimal solution of (2).

After we have discovered the optimal solution $\bar{y}$ of problem (2) we need to reconstruct $\left\{x(t),u(t)\right\}_{t=0}^{T-1}$, the optimal solution of problem (1). Let $J = \left\{j \mid \lambda_j^m > 0\right\}$. Then $\bar{y} = \sum_J \lambda_j^m y^i$. For $j \in J$, let $\pi(t,j)$ be the appointment policy used to generate y^j. To go from x to y^j we hire only in rank $\pi(t,j)$ at time t. If $\left\{x^j(t),u^j(t)\right\}_{t=0}^{T-1}$ leads from x to y^j at cost W_i then $\left\{x(t),u(t)\right\}_{t=0}^{T-1}$ leads from x to $\bar{y}$ at cost $\sum_{i=1}^{m} \lambda_i^m W_i$, where

$$u(t) = \sum_J \lambda_j^m u^j(t)$$

$$x(t) = \sum_J \lambda_j^m x^j(t) .$$

The justification of our algorithm is as follows. Let $\bar{y}$ be the optimal solution of (2). For any m, if λ^m is optimal for (3), then $\sum_{i=1}^{m} \lambda_i^m y^i$ is feasible for (2). It is easy to show that $W(y)$ is convex, hence

$$W(\bar{y}) \leq W\left(\sum_{i=1}^{m} \lambda_i^m y^i\right) \leq \sum_{i=1}^{m} \lambda_i^m W_i .$$

In addition the duality Theorem of linear programming implies $r^m \geq 0$ and,

$$W(\bar{y}) \leq \sigma^m = \sum_{i=1}^{m} \lambda_i^m W_i .$$

Thus σ^m is an upper bound on the optimal value of (2). Since $r^m \geq 0$, and $\bar{y}A \geq 0$ we have $\bar{y}Ar^m \geq 0$. In addition $\bar{y}$ is feasible for (4) so

$$\nu^{m+1} = \text{Min } W(y) - yAr^m \leq W(\bar{y}) - \bar{y}Ar^m \leq W(\bar{y}) .$$

Therefore ν^{m+1} is a lower bound on the optimal value $W(\bar{y})$ of (2).

The stopping condition, $\nu^{m+1} \geq \sigma^m$, implies our current lower bound equals the upper bound, and we have found the optimal solution.

The proof is completed by showing problem (4) can be solved by setting $q = Ar^m$ and using the dynamic programming procedure developed in Section I. Let ν^{m+1} and y^{m+1} be the value and terminal point from the dynamic programming calculation. Suppose there exists a y such that $W(y) - yAr^m < \nu^{m+1}$. Then the program $\left\{x(t), u(t)\right\}_{t=0}^{T-1}$ leading to y would be optimal for the dynamic program. This contradicts the definition of ν^{m+1}. It is also apparent that $\nu^{m+1} = W(y^{m+1}) - y^{m+1}Ar^m$. If there is a cheaper way to y^{m+1} we could do better than ν^{m+1} in the dynamic program. Note the dynamic program always generates an extreme point of $R^T(x)$, since it appoints in only one rank in each period. $R^T(x)$ has a finite number of extreme points, therefore we must converge to an optimal solution in a finite number of steps. This completes our discussion of the generalized linear program.

The feasibility of (1) is checked in a similar way. First set $q = 0$, then find a $y^1 \in R^T(x)$ and $W_1 = W(y_1) = \text{Min } W(y)$. For any m solve a *phase I master*.

(5) $$\text{Minimize} \quad \sum_{k=1}^{K} z_k$$

$$\text{Subject to} \quad \sum_{i=1}^{m} \lambda_i (y_i A) + z \underset{=}{>} 0$$

$$\sum_{i=1}^{m} \lambda_i = 1$$

$$\lambda \underset{=}{>} 0 \,, \qquad z \underset{=}{>} 0 \,.$$

Let (λ^m, z^m) solve (5). If $\sum_{k=1}^{K} z_k^m = 0$ then $\sum_{i=1}^{m} \lambda_i y_i$ is feasible for (2). Otherwise let (r^m, σ^m) be the optimal dual solution of (5) and solve the dynamic program in Section I *with* $c(t) = d(t) = 0$ *for all* t and $q = Ar^m$.

Let y^{m+1} be the terminal point of the dynamic program. If $\sigma^m > - y^{m+1} A r^m$, then add column $(0, y^{m+1} A)$ to problem (5). If $\sigma^m \underset{=}{<} - y^{m+1} A r^m$ then stop: *problem (2) is infeasible.*

The justification is similar to the phase II problem described previously. When switching from phase I to phase II we need to know W_i for $i = 1, 2, \ldots, m$. When we solve the dynamic programming subproblem in phase I we should calculate the cost of going from x to y^i using the optimal program. This is an upper bound (usually exact) on W_i. A careful review of the phase II procedure should convince the reader that it is possible to substitute these bounds for the W_i and commence phase II directly. If one of the bounds is not tight, then the column will be regenerated if it is needed in the course of calculations.

Problem (1) can be considered as a linear program with $T + n$ equality constraints and $n \times (T + 1)$ nonnegative variables. In contrast problem (3) has $n + 1$ equality constraints and $n + M$ variables, where M is the number of extreme points of $R^T(x)$. When $T = 30$, and $n = 15$, then (1) has 45 constraints to 16 constraints in (3).

We elected to solve problem (3) for several reasons: the efficiency of the dynamic programming subroutine, the reduced storage, the ease of handling arrays, and the flexibility in changing the data.

III. INFINITE HORIZON PROBLEMS

This section considers stationary problems with an infinite planning horizon and with constraints on the staff distributions in each period. In practice most planning problems do not have a specified finite horizon; many manpower policy decisions have such long term effects that decisionmakers must consider the long run implications of any policy. One way to examine the long run effects of present decisions is to select a large planning horizon T with the hope that the impact of present decisions on periods $t \geq T$ is likely to be small. This approach has two drawbacks: it is difficult to select an appropriate T, and if T is large the resulting multi-stage planning problem is difficult to solve. Our alternative approach is to assume an infinite planning horizon in a stationary environment. Thus, with the exception of the state vector x, every period we begin a new problem that is identical to the problem we faced at time zero. These stationary problems are frequently easier to solve than long finite horizon problems.

In each time period the system will obey the law of motion, the no firing condition, and the growth constraint. In addition we shall require that for each t the staff distribution $x(t)$ is contained in the polyhedral cone $C = \{y \mid yA \geq 0\}$.

Theorem 2 describes theoretical conditions necessary and sufficient for a sequence of staff distributions to remain in C indefinitely. From these conditions we develop in Theorem 3 a test for the initial staff distribution which will guarantee the existence of feasible solutions to the infinite horizon minimum cost problem. Two objective criteria are considered--long run average cost and total discounted cost, and for both cases bounds on the minimum cost have been calculated. These results, which are contained in Theorems 4 and 5 respectively, were accomplished with the aid of a surrogate linear program. The surrogate program defines a stationary linear appointment policy $u(t) = x(t)D$

which obeys the law of motion, no firing, and growth constraints. Given an initial distribution $x(0)$, the appointment policy $u = xD$ determines the entire sequence $x(t)$. If $x(t) \in C$ for all t then the stationary linear appointment policy is optimal and the surrogate linear program's lower bound is exact. It is, however, possible that $x(t) \notin C$ for some t. In this case the lower bound is not exact. In any case we have $\frac{x(t)}{x(t)f} \to y^* \in C$ as $t \to 0$, and in addition a weighted average (which depends on the discount factor) of the $x(t)$ is in C. These properties lead us to suspect that, even if $x(t) \notin C$ for some t, our lower bound is accurate and our easily implemented appointment policy is nearly optimal.

As we point out at the close of this section it is theoretically possible to find the actual optimal policy. However it is a difficult task and not a task we would like to repeat to gain sensitivity information. What is more, it will not in general lead to an easily implemented linear decision rule.

The infinite horizon constraints are:

$$
\begin{aligned}
& x(t) = x(t-1)Q + u(t-1) \\
(1) \qquad & x(t)f = \delta^t xf \\
& u(t-1) \geq 0 \\
& x(t)A \geq 0 \qquad t = 0,1,2,\ldots \\
& x(0) = x \geq 0 \qquad \text{given.}
\end{aligned}
$$

The use of Q and δ above instead of P and θ as before is in anticipation of Section IV, where different policies will be considered for promotion and growth in the transient and stationary stages respectively.

The first three constraints enforce the law of motion, the constant growth rate of $x(t)f$, and the no firing condition. The final constraint restricts

the distribution $x(t)$ to the cone $C = \{y \mid yA \geqq 0\}$ for all t.

The constraints can be written in terms of $x(t)$ alone.

$$
\begin{aligned}
& x(t) \geqq x(t-1)Q \\
(2) \qquad & x(t)f = \delta^t xf \\
& x(t)A \geqq 0 \qquad t = 1,2,\ldots \\
& x(0) = \geq 0 \qquad \text{given}
\end{aligned}
$$

We can also write the constraints in terms of $u(t)$, since

$$x(t+1) = xQ^{t+1} + \sum_{s=0}^{t} u(s)Q^{t-s} .$$

We have

$$
\begin{aligned}
(3) \qquad & \sum_{s=0}^{t} u(s)Q^{t-s}f = x[\delta^{t+1}I - Q^{t+1}]f \\
& \sum_{s=0}^{t} u(s)Q^{t-s}A \geqq - xQ^{t+1}A \\
& u(t) \geqq 0 \qquad t = 0,1,2,\ldots
\end{aligned}
$$

The objective is to minimize $\sum_{t=0}^{\infty} \alpha^t[x(t)c + u(t)d]$. When $\alpha\cdot\delta < 1$, this sum is finite. If $\alpha\cdot\delta = 1$, the cost is typically infinite. In this case we seek to minimize

$$(4) \qquad \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \alpha^t[x(t)c + u(t)d] .$$

Before searching for optimal policies we must examine the question of feasibility. Given $x \in C$ is it possible to remain in C indefinitely? This question cannot be answered precisely. However, we do obtain a sufficient condition for $x \in C^{\infty}$ in Theorem 3, and a necessary condition from Theorem 5. Both conditions require the solution of a linear program.

Recall from the introduction that $E \triangleq \{y \mid \delta y \geq yQ, y \geq 0\}$. The following lemma is extremely useful in analyzing questions of feasibility.

<u>Lemma 1</u>:

If $\left\{x(t)\right\}_{t=0}^{\infty}$ is feasible for (2), define.

$$y = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \delta^{-t} x(t)$$

then $y \in C \cap S \cap E$.

<u>Proof</u>:

For each T let

$$y(T) = \frac{1}{T} \sum_{t=0}^{T-1} \delta^{-t} x(t) .$$

$y(T)$ is a nonnegative combination of points in C , therefore $y(T)$ is in C . For each t , $\delta^{-t}x(t) \in S = \{z \mid zf = xf, z \geqq 0\}$, thus $y(T) \in S$, since $y(T)$ is a convex combination of points in S . It is apparent that $y \in C \cap S$.

From the constraints of (2), $\delta^{-t}x(t) \geqq \frac{1}{\delta}[\delta^{-(t-1)}x(t-1)Q]$. Summing the above relation for $t = 0, \ldots, T-1$ (with $x(-1) = 0$) , we obtain

$$y(T) \geqq \left[\frac{1}{\delta} y(T)Q - \frac{\delta^{-(T-1)}x(T-1)Q}{T}\right].$$

Now $\delta^{-(T-1)}x(T-1)Q$ remains bounded, therefore as $T \to \infty$, $y \geqq \frac{1}{\delta} yQ$. Also,

since $\delta^{-t}x(t) \leq xf(\min_i f_i)^{-1}e$ for each t, we have $y < \infty$. Thus $y \varepsilon E$. ||

Recall $R(x) = \{y \mid y \geq xQ, yf = \delta xf\}$. Let $C^0 = C$, and for $t = 1,2, \ldots,$ define

$$C^t = \{x \mid x \varepsilon C \text{ and } R(x) \cap C^{t-1} \neq \phi\} .$$

C^t is the set of starting points x such that it is possible to remain in C for t steps. Clearly $C^t \supseteq C^{t+1}$. Let $C^\infty = \bigcap_{t=0}^{\infty} C^t$. Then C^∞ is the set of feasible starting points for problem (1). It follows that

<u>Lemma 2</u>:

$x \varepsilon C^\infty$ if and only if $x \varepsilon C$ and $R(x) \cap C^\infty \neq \phi$.

Theorem 2 below gives an operational check on whether feasible initial states exist. It does not answer the more difficult question of whether a particular starting state x is in C^∞.

<u>Theorem 2</u>:

C^∞ is void if and only if $C \cap E$ is void.

<u>Proof</u>:

If $x \varepsilon C \cap E$, then $x \varepsilon C^\infty$ since $\delta^t x$ is feasible for (2).

For the converse, suppose $x \varepsilon C^\infty$. From Lemma 2 we can construct a sequence $\{x(t)\}_{t=0}^{\infty}$ such that $x(0) = x$ and $x(t+1) \varepsilon R[x(t)] \cap C^\infty$ for $t = 0,1, \ldots$. From Lemma 1 we can then find a y such that $y \varepsilon C \cap E$. ||

As mentioned above these results do not give an explicit characterization of C^∞, the set of feasible starting points. In some cases it is possible to prove that $C^\infty = C$. Let $\{r_k\}_{k=1}^{K}$ be the extreme rays of the cone C. Then

$$C = \left\{y \mid y = \sum_{k=1}^{K} \mu_k r_k, \mu_k \geq 0\right\} .$$

If $R(r_k) \cap C \neq \phi$ for all k, then $R(y) \cap C \neq \phi$ for each $y \in C$. Therefore $C^1 = C$, a condition which leads to the identity $C^\infty = C$. In Theorem 3 we apply this result to develop a means of testing whether a given value of $x \in C$ is a member of C^∞.

<u>Theorem 3</u>:

If $x \in C$, then $x \in C^\infty$ if the linear program (5), below, has an optimal solution $\lambda^* < 1$.

$$\text{(5)} \qquad \text{Minimize } \lambda$$

$$\begin{aligned} \text{Subject to } \lambda x + u - y &= 0 \\ y\tilde{f} &= \delta x f \\ y &\geq xQ \\ uA &\geq 0 \\ u(\delta I - Q) &\geq 0 \\ \lambda \geq 0,\ u \geq 0,\ y &\geq 0 . \end{aligned}$$

<u>Proof</u>:

The constraints of (5) require that $u \in C \cap E$ and that $y \in R(x)$. Let (λ^*, u^*, y^*) solve (5), with $\lambda^* < 1$. Then define $\tilde{u}$ to satisfy $(1 - \lambda^*)\tilde{u} = u^*$. Obviously $\tilde{u} \in C \cap E$ and $y^* = \lambda^* x + (1 - \lambda^*)\tilde{u}$. Now consider a new problem with $\tilde{C} = \{y \mid y = \mu_1 x + \mu_2 \tilde{u}, \mu_1 \geq 0, \mu_2 \geq 0\}$. Since x and $\delta^{-1}\tilde{u}$ are in C, we have $\tilde{C} \subseteq C$, and $\tilde{C}^\infty \subseteq C^\infty$. However $y^* \in \tilde{C} \cap R(x)$ and $\delta\tilde{u} \in \tilde{C} \cap R(\tilde{u})$, therefore $\tilde{C}^\infty = \tilde{C}^1 = \tilde{C}$, and $x \in \tilde{C}^\infty \subseteq C^\infty$. ||

Note that problem (5) contains $3 \times N + K + 1$ constraints with $2N + 1$

nonnegative variables. The dual problem of (5) will have only $2N + 1$ constraints and thus will be easier to solve.

Now we turn to optimization and consider the average cost problem $(\alpha\delta = 1)$. If we substitute $u(t) = x(t+1) - x(t)Q$ in (4) we obtain

$$\text{Minimize} \quad \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \alpha^t x(t)g \; .$$

$$\text{Subject to} \quad x(t) \geqq x(t-1)Q \; ,$$

(6)
$$x(t)f = \delta^t x f \quad ,$$

$$x(t)A \geqq 0 \quad , \quad t = 0,1,2, \ldots$$

$$x(0) = x \geq 0 \quad \text{given}$$

where $g = c + \delta d - Qd$. Since $\delta^{-t} = \alpha^t$, we know from Lemma 1 that the value of any solution $\left\{x(t)\right\}_{t=0}^{\infty}$ is given by yg where

$$y = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \delta^{-t} x(t) ,$$

and $y \in C \cap S \cap E$. We can therefore calculate a *lower* bound on the value of (6) by solving:

(7)
$$\begin{aligned} &\text{Minimize} \quad yg \\ &\text{Subject to} \quad y \geqq y\,Q \\ &\qquad yf = xf \\ &\qquad yA \geqq 0 \\ &\qquad y \geqq 0 \end{aligned}$$

Problem (7) can be written in a more operational form. Since, $y = y\alpha Q + u$, we set $y = u(I - \alpha Q)^{-1} = uB$ where $B \geqq 0$. Problem (7) becomes;

(8)
$$\begin{aligned} &\text{Minimize} \quad uBg \\ &\text{Subject to} \quad uBf = xf \\ &\qquad uBA \geqq 0 \\ &\qquad u \geqq 0 . \end{aligned}$$

<u>Theorem 4</u>:

If $v = [\delta I - Q]f > 0$ and $\alpha\delta = 1$, then

(i) The optimal value of (8) is a lower bound on the optimal value of (6).

(ii) If we maximize in (8) we obtain an upper bound on the optimal value of (6).

(iii) If x is in the relative interior of $E \cap C$, then the bound obtained in (8) is tight and an optimal policy can be described.

(iv) If u^* is the optimal solution of (8) then the stationary linear appointment policy is

$$u(t) = x(t) \frac{vu^*}{u^*f} = x(t)D$$

<u>Proof</u>:

Items (i) and (ii) follow from the discussion above and Lemma 2.

We shall sketch a proof of item (iii). Let u^* solve (8), $y^* = u^*B$, and let L be the line segment joining x and $y^* \varepsilon S$. For each t let $z(t) = \alpha^t x(t) \varepsilon S$. Given $z(t)$ the optimal policy is to choose $z(t + 1)$ as the point in $L \cap R[\alpha z(t)]$ that is closest to y^*. This rule generates a sequence of points $z(t) \varepsilon C$ that converges to y^*.

Item (iv) warrants more discussion than proof. First, note that v is a column vector and u^* is a row vector, therefore vu^* is an $n \times n$ matrix and u^*f is a scalar. We can easily verify that this appointment policy leads to a sequence $x(t)$ that satisfies

$$x(t) \geqq x(t - 1)Q$$

and

$$x(t)f = \delta^t xf$$

for all t. In addition let $z(t) = \alpha^t x(t)$. Since $x(t + 1) = x(t)[Q + D]$ we have $z(t + 1) = z(t)[\alpha(Q + D)] = z(t)G$ where $z(t) \varepsilon S$ for all t. It is easy to see that $z(t) = xG^t$.

G is a square nonnegative matrix. The vectors f and y^* are easily seen to be right and left eigenvectors of G corresponding to an eigenvalue

of 1 . Let F be a diagonal matrix with elements f_i on the diagonal. Then $F^{-1}GF$ is a Markov matrix. We shall assume that this Markov matrix corresponds to a chain with a single ergodic class ([5], page 69). This condition is satisfied if, for example, $u_1^* > 0$ and it is possible to reach any rank starting at rank 1. Let π be the unique probability vector ([5], page 71), that solves $\pi = \pi F^{-1}GF$. One can check that $\pi_i = f_i y_i^*$.

In addition $(F^{-1}GF)^t \to H$ where each row of H is equal to π . Then $G^t = F[F^{-1}GF]^t F^{-1} \to FHF^{-1}$. Thus $z(t) = xG^t \to xFHF^{-1} = y^*$.

This establishes two facts. The stationary linear policy achieves the lower bound determined by (8) and leads to a sequence of states $x(t)$ such that

$$\lim_{t\to\infty} \frac{x(t)}{x(t)f} \in C . ||$$

It is a conjecture that the optimal value function of problem (6) is constant for initial states x in the relative interior of C^∞ . It is possible to get stuck on the boundary of C^∞ and thus incur a higher per period cost. Note that if the conjecture above is true it is not operational since it requires a precise characterization of C^∞ and the specification of an optimal appointment policy.

We now turn our attention to the problem of minimizing discounted cost. In the discounted case we shall frequently use the identity

$$\sum_{t=0}^{\infty} \alpha^t x(t) = xB + \alpha \sum_{t=0}^{\infty} \alpha^t u(t)B$$

where $B = (I - \alpha Q)^{-1}$. Using this identity with constraints (3) and the discounted cost objective we obtain

(13) $$\text{Minimize} \quad xBc + \sum_{t=0}^{\infty} \alpha^t u(t)[\alpha Bc + d]$$

$$\text{Subject to} \quad \sum_{s=0}^{t-1} u(s)Q^{t-s-1}f = x[\delta^t I - Q^t]f$$

$$\sum_{s=0}^{t-1} u(s)Q^{t-s-1}A \geqq - xQ^t A$$

$$u(t) \geqq 0 \qquad t = 0,1,2, \ldots$$

Define $u = \sum_{t=0}^{\infty} \alpha^t u(t)$. Then $\sum_{t=0}^{\infty} \alpha^t x(t) = xB + \alpha uB$, and the objective value of any solution in (9) is

$$xBc + u[\alpha Bc + d] \ .$$

We wish to substitute a finite set of constraints for the infinite set (9). Instead of writing

$$\sum_{x=0}^{t-1} u(s)Q^{t-s-1}A \geqq - xQ^t A$$

for all $t = 0, \ldots$, we multiply the t^{th} constraint by α^t and sum. The resulting problem is

(10) $$\text{Minimize} \quad u[\alpha Bc + d] + xBc$$

$$\text{Subject to} \quad u[\alpha Bf] = \frac{xf}{1 - \alpha\delta} - xBf$$

$$u[\alpha BA] \geqq - xBA$$

$$u \geqq 0$$

This problem has $K + 1$ constraints, within nonnegative variables. We know that any feasible solution of (9) determines a feasible solution of (10).

Theorem 5:

If $v = [\delta I - Q]f > 0$, and $\alpha\delta < 1$, then

(i) Infeasibility of (10) implies $x \notin C^{\infty}$.

(ii) If u^* solves (10), then $u^*[\alpha Bc + d] + xBc$ is a lower bound on the optimal value of (9).

(iii) The stationary linear appointment policy is

$$u(t) = x(t)\left[\frac{v\,u^*}{u^* f}\right] = x(t)D \ .$$

Proof:

Parts (i) and (ii) follow since any $\left\{u(t)\right\}_{t=0}^{\infty}$ feasible in (9) implies $u = \sum_{t=0}^{\infty} \alpha^t u(t)$ is feasible in (10).

In part (iii) it is easy to verify that the stationary linear decision rule leads to a sequence of distributions $x(t)$ which satisfy the law of motion and growth constraint. As before we can also show that in the limit $\frac{x(t)}{x(t)f}$ is in C. In addition $\sum_{t=0}^{\infty} \alpha^t x(t)A \geq 0$. Let $u(t)$ be the appointment policy resulting from the stationary linear appointment rule. Note that

$$u = \sum_{t=0}^{\infty} \alpha^t u(t) = \frac{1}{u^* f}\left[\sum_{t=0}^{\infty} \alpha^t x(t)v\right] u^* = \eta u^*$$

where η is a constant. Recall that the sequence $u(t)$ is designed so that $u(t)f = \delta^t xf$ for all t. Therefore u is feasible for (10) and

$u[\alpha Bf] = \eta u^*[\alpha Bf] = 1$. However, u^* is feasible for (9), thus $u^*[\alpha Bf] = 1$, so $\eta = 1$ and $u = u^*$. Thus the stationary linear policy *achieves* the lower bound calculated by (10).||

We say a distribution of staff y^* is stationary if the stationary linear decision rule $u(t) = x(t)D$ applied at $x(0) = y^*$ leads to the sequence $x(t) = \delta^t y^*$.

In the average cost case the stationary staff distribution y^* is given by $y^* = u^*B$, and the optimal cost by y^*g . In the average cost case to find the stationary staff distribution we must solve.

$$\delta y^* = y^*\left[Q + \frac{vu^*}{u^*f}\right]$$

$$y^*f = xf$$

This system of equations is similar to the characterization of the limiting distribution in a finite, regular, Markov chain [5]. The system can be solved in the same manner.

As a final comment let $W(x)$ be the optimal value of problem (9) and $V(x)$ the optimal value of problem (10) . We know $V(x) \leq W(x)$, and $W(x)$ satisfies a functional equation

$$W(x) = \min_{\substack{u \geq 0 \\ uf = xv \\ uA \geq -xQA}} \left[xc + du + \alpha W[xQ + u]\right] ,$$

where $W(x) = +\infty$ if $x \notin C^\infty$. It is theoretically possible using a generalization of Howard's policy iteration scheme (see [9]) to compute $W(x)$ for $x \in S$ and $u(x)$ an optimal policy. We suspect that $W(x)$ and $u(x)$ are piecewise linear for $x \in C^\infty$.

IV. TRANSIENT-STATIONARY PROBLEMS

New institutions or institutions experiencing severe change often undergo a transient period before reaching a state of equilibrium growth. This section shows how the infinite horizon model of Section III can be combined with results from Sections I and II to calculate lower bounds on the minimum cost of the combined transient and stationary problem.

In a typical transient problem the system grows rapidly during the first T years at 10% per year. After reaching maturity the system may grow at a reduced rate of 1% per year. In this section we provide a method for analyzing transient problems which will allow a decision maker to gauge the tradeoffs between the initial growth rate and the duration of the transient period.

Let P be the promotion matrix during the transition years and Q the promotion matrix during the steady state period. P and Q may not be equal because of different promotion policies in the two periods. Let θ and δ be the growth rates during the different periods.

Now consider a problem with a T year transient period, and assume no restriction is placed on the distribution $x(t)$ during the transient stage. In the steady state from period T onward we impose the restriction $x(t) \in C = \{y \mid yA \geqq 0\}$ for $t \geqq T$.

As in Section II, let $W[y]$ be the minimum cost of going from x to y in T periods. Let $V[y]$ be the optimal value of problem (10) in Section III. $V[y]$ is a lower bound on the minimum cost operating the system over an infinite horizon with initial state y.

$$V(y) = yBc + \text{Min } u[\alpha Bc + d]$$

$$\text{Subject to } \quad u[\alpha Bf] = \frac{yf}{1 - \alpha\delta} - yBf$$

$$u[\alpha BA] \geqq -yBA$$

$$u \geqq 0$$

To calculate a lower bound on the transient and stationary phase of operation we must solve

(2) $$\text{Minimize} \quad W(y) + \alpha^T V(y)$$

$$\text{Subject to} \quad y \in R^T(x) \cap C \ .$$

We outline a procedure for solving (2) below. The method follows the generalized programming technique of Section II. It is important to find a technique that will produce an optimal or nearly optimal solution of (2) efficiently. The efficiency in solution allows one to solve (2) repeatedly and to carry out sensitivity analyses.

The master program is:

(3) $$\text{Minimize} \quad \sum_{i=1}^{m} \lambda_i(W_i + y_i\alpha^T Bc) + u[\alpha^{T+1}Bc + \alpha^T d]$$

Subject to:

$$\sum_{i=1}^{m} \lambda_i y_i A \geqq 0$$

$$\sum_{i=1}^{m} \lambda_i = 1$$

$$\sum_{i=1}^{m} \lambda_i y_i BA + u\alpha BA \geqq 0$$

$$\sum_{i=1}^{m} \lambda_i(y_i Bf) + u(\alpha Bf) = \frac{\theta^T xf}{1 - \alpha\delta} = \rho$$

$$\lambda_i \geqq 0 \qquad u \geqq 0$$

where the y_i are points in $R_T(x)$ and $W_i = W(y_i)$.

Let (λ^m, u^m) solve (3) and let $(r^m, \sigma^m, s^m, \psi^m)$ be the optimal dual solution.

We have:

$$(4)\quad (i)\quad (\alpha BA)s^m + (\alpha Bf)\psi^m \leqq (\alpha^T)(\alpha Bc + d)$$

$$(ii)\quad (y_iA)r^m + \sigma^m + (y_iBA)s^m + (y_iBf)\psi^m \leqq (W_i + y_i\alpha^T Bc)$$

for $i = 1,2, \ldots, m$

$$(iii)\quad r^m \geqq 0,\ s^m \geqq 0$$

$$(iv)\quad \sigma^m + p\psi^m = \sum_{i=1}^{m} \lambda_i^m (W_i + y_i\alpha^T Bc) + u^m[(\alpha^T)(\alpha Bc + d)]\ .$$

The first three conditions state that $(r^m, \sigma^m, s^m, \psi^m)$ is dual feasible. The final condition is that both solutions are optimal.

As in Section II, $\sigma^m + \psi^m p$ provides an upper bound on the optimal value of (2). The optimality of our current solution, $\sum_{i=1}^{m} \lambda_i^m y_i$, is checked using the subproblem:

$$(5)\qquad \text{Minimize}\qquad W(y) - y[Ar^m + BAs^m + Bf\psi^m - \alpha^T Bc]$$

$$\text{Subject to}\qquad y \in R^T(x)$$

The optimal solution of (5) is calculated using the dynamic program with $q^m = Ar^m + BAs^m + Bf\psi^m - \alpha^T Bc$. Let ν^{m+1} be the optimal value of (5) and y^{m+1} the terminal point.

The value ν^{m+1} can be used to calculate a lower bound on the optimal value of (2). Let V^* be the optimal value of (2), then

$$\sigma^m + \psi^m p \geqq V^* \geqq \nu^{m+1} + \psi^m p$$

There are two cases to consider:

1. If $\nu^{m+1} \geqq \sigma^m$, then the current solution $\bar{y} = \sum_{i=1}^{m} \lambda_i^m y_i$ solves (2).
2. If $\nu^{m+1} < \sigma^m$, then the column $(W_{m+1} + \alpha^T y^{m+1} Bc,\ y^{m+1}A,\ 1,\ y^{m+1}BA,\ y^{m+1}Bf)$ can enter the basis in (3).

To verify the lower bound claim consider an alternate way of expressing $\alpha^T V(y)$ using the duality theorem.

$$\alpha^T V(y) = \alpha^T yBc + \text{Max} \quad \psi(p - yBf) - yBAs$$

(5) $$\text{Subject to} \quad (\alpha Bf)\psi + (\alpha BA)s \leq \alpha^T(\alpha Bc + d)$$

$$s \geq 0$$

According to 4(i) ψ^m and s^m are feasible for (5), thus:

(6) $$\alpha^t V(y) \geq \alpha^T yBc + \psi^m(p - yBf) - yBAs^m$$

for all y. In particular for $\bar{y}$, the optimal solution of (2).

Since the optimal solution $\bar{y} \varepsilon C$, and $r^m \geq 0$, we have $\bar{y}Ar^m \geq 0$. Thus

(7) $$(\bar{y}) \geq W(\bar{y}) - \bar{y}Ar^m .$$

Combining (6), with $y = \bar{y}$, and (7) we obtain

$$V^* = W(\bar{y}) + \alpha^T V(\bar{y}) \geq W(\bar{y}) - \bar{y}[Ar^m + BAs^m + Bf\psi^m - \alpha^T Bc] + \psi^m p$$

$$W(y^-) - y^-[q^m] + \psi^m p = v^{m+1} + \psi^m p$$

We can use the technique described in Section II, to obtain a first feasible solution to problem (3).

It is possible to reformulate (2) as a linear program with $T + 2 \times n$ constraints and $n \times (T + 2)$ variables. For reasons identical to those given in Section II and the introduction, we preferred the generalized linear programming approach.

V. USE OF THE MODEL

Our model was originally motivated by an examination of a university faculty system. This section demonstrates how the model can be applied to a faculty system with three ranks of members: assistant professors, associate professors, and full professors. This highly aggregated example was selected to facilitate geometrical display of the results and more importantly, to stress our contention that this type of model is best suited to evaluate and suggest general policies or to distinguish the effects of alternate policies. We do not view our model as a precise control instrument. In the university context: we are not trying to specify appointment quotas in each department for each year, rather we are concerned with judging the long range financial and staffing effects of alternate promotion, salary, and retirement policies.

A small model has the advantage of flexibility in altering the structural coefficients. For example, the promotion rates, P_{ij} , are actually the result of policy decisions. If a change in the promotion policy is being considered, it is relatively easy to alter the P_{ij} and, resolve the problem, and judge the impact of the new promotion policy. A more disaggregated model might represent the dynamics of the system more accurately, but we rapidly lose flexibility as the number of ranks increases.

The algorithms were programmed in APL 360 with the aid of Edward Stohr and then used to solve two classes of problems.

To begin, we examined the problem of reaching a specified distribution of faculty in 15 years at minimum cost. This problem was solved under four separate assumptions concerning promotion and retention policy and salary structure. In each of the four cases we varied the target distribution over a wide range of possibilities. Our computational efforts involved about 150 of the generalized linear programs described in Section II. Since each generalized linear program used the dynamic programming subroutine about 14 times,this calculation included

solving 2000 of the dynamic programs described in Section I. We shall present one set of results in detail and comment on others.

The following data were used.*

Initial distribution: $x_1 = .3$, $x_2 = .3$, $x_3 = .4$

Discount factor 1: i.e. total cost.

Growth rate 0 (constant size).

Support cost (20,28,34)

Recruiting and hiring cost (2,2,2)

Time horizon: 15 years.

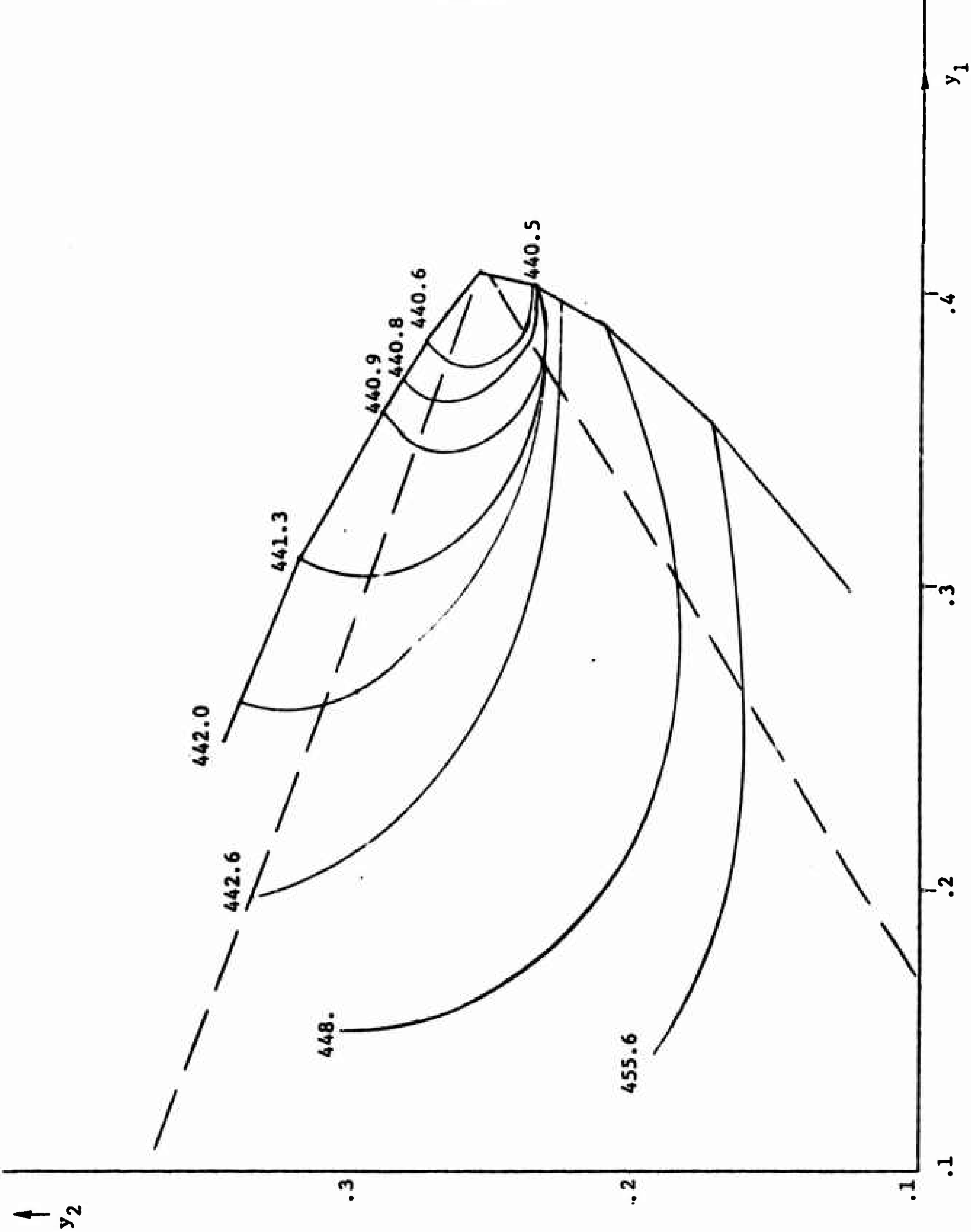
Promotion Matrix

$$P = \begin{bmatrix} .71 & .12 & 0 \\ & .8 & .1 \\ & & .93 \end{bmatrix}$$

In Figure 1 below, y_1 and y_2 refer to the target values of assistant and associate professors. Obviously, the fraction of full professors is given by $y_3 = 1 - y_1 - y_2$. The dark border shows the boundary of distributions that can be obtained from x in 15 periods $(R_{15}(x))$. The dotted straight lines indicate the boundary of the equilibrium set. If a distribution y is in the equilibrium set, then there exists a hiring policy such that the distribution y can be maintained from year to year. The curved lines in Figure 1 are isocost lines: any two targets on the same isocost line can be reached for the same minimum cost. Note the isocost lines as drawn do not differ by a constant amount. The cost function is actually bowl shaped. It is fairly flat over a large region, and has relatively steep sides. In particular, as y_3 increases, the cost increases sharply. Note also that the cost increases when the target point is outside of the equilibrium set. This is due to the difficulty the system has in reaching a nonequilibrium target.

*One should not attach undue importance to these figures; our purpose is to give an example of the models' usefulness and not to solve a particular problem at a particular university.

FIGURE 1



The numbers on the isocost lines are the cost per faculty member in thousands of dollars of running the system for 15 years. To estimate the total cost you must multiply by the number of faculty members.

The costs of reaching various target points were calculated under three alternative set promotion and salary policies. First we changed the first top row of the promotion matrix to .666 .1666 .0 . This change had two effects: it increased costs by about 3%, and it reduced the size of the feasible region. Targets with $y_1 \geq .34$ became infeasible. Thus a more liberal promotion policy did not cause a drastic increase in costs, however it did cause the loss of a great amount of flexibility. In the next trial we moved to a more stringent promotion policy and changed the top row of the promotion matrix to .65 .08 .0 . This resulted in a 2% decrease in costs and an enlargement of the set of feasible target. The final application retained the more stringent promotion policy and raised the support cost of full professors to 40 . This, of course, did not change the set of feasible targets, but it did cause an 8% increase in cost.

The second major application used the transient-infinite horizon model described in Section IV. We consider the problem of an institution that wishes to expand its faculty size by 500%. The model is used to measure the tradeoff between cost and the length of the growth period. We checked growth periods from 8 to 24 years, and plotted cost vs. growth period in the curve shown below.

The cost figures have been normalized so that they show the percent of saving that is achieved by lengthening the growth period. The cost calculated is a lower bound on the discounted cost of going from an initial faculty distribution to a target distribution (5 times as large) in T years and then remaining at that target indefinitely.

In the example solved, the following data were used:

Initial distribution: (.3 .3 .4)

Target distribution: (.31 .275 .415)

Target size: five times original size-geometric growth.

Discount factor: $\alpha = .96$

Support costs: (20 28 34)

Appointment costs: (2 2 2)

Promotion matrix in the growth stage

$$\begin{bmatrix} .71 & .12 & .0 \\ & .8 & .1 \\ & & .93 \end{bmatrix}$$

Promotion matrix in the steady state.

$$\begin{bmatrix} .65 & .08 & 0 \\ & .8 & .1 \\ & & .93 \end{bmatrix}$$

The growth rate, of course, is adjusted to meet the target size in the specified time period. The coefficients are somewhat arbitrary, particularly the discount rate. The estimates of support and appointment cost, and the promotion matrices, as noted above, contain the result of policy decisions. The small size of this model allows one to test sensitivity to changes in the discount factor, salary or promotion policy.

Although our three state model is crude, it must be judged against past performance of faculty planning in major universities. Some universities have been through expansion periods in the post-war era without considering the problems that would ensue when the growth period was over. A naive shortsighted strategy produced a nonequilibrium result. This is akin to impacting astronauts on the moon rather than landing them there. In the first instance you attain your target with catastrophic results. In the second case a smooth transition to equilibrium is assured. Our model does produce a smooth transition to

equilibrium. In addition, we can determine a least cost path to the equilibrium distribution.

Another possible use of the model is in determination of the minimum time needed to reach a given distribution. For this problem the final distribution $x(T)$ is fixed and T is then reduced until the problem becomes infeasible. This approach not only yields the minimum number of periods until $x(T)$ can be reached but it gives a tradeoff curve relating the number of periods and the cost of reaching the target. If the model is of reasonable size there will be little computational difficulty in this approach, and a decision maker could resolve the same minimum time problem under various assumptions on the promotion matrix to gauge the effects of promotion policy on the time to reach a target.

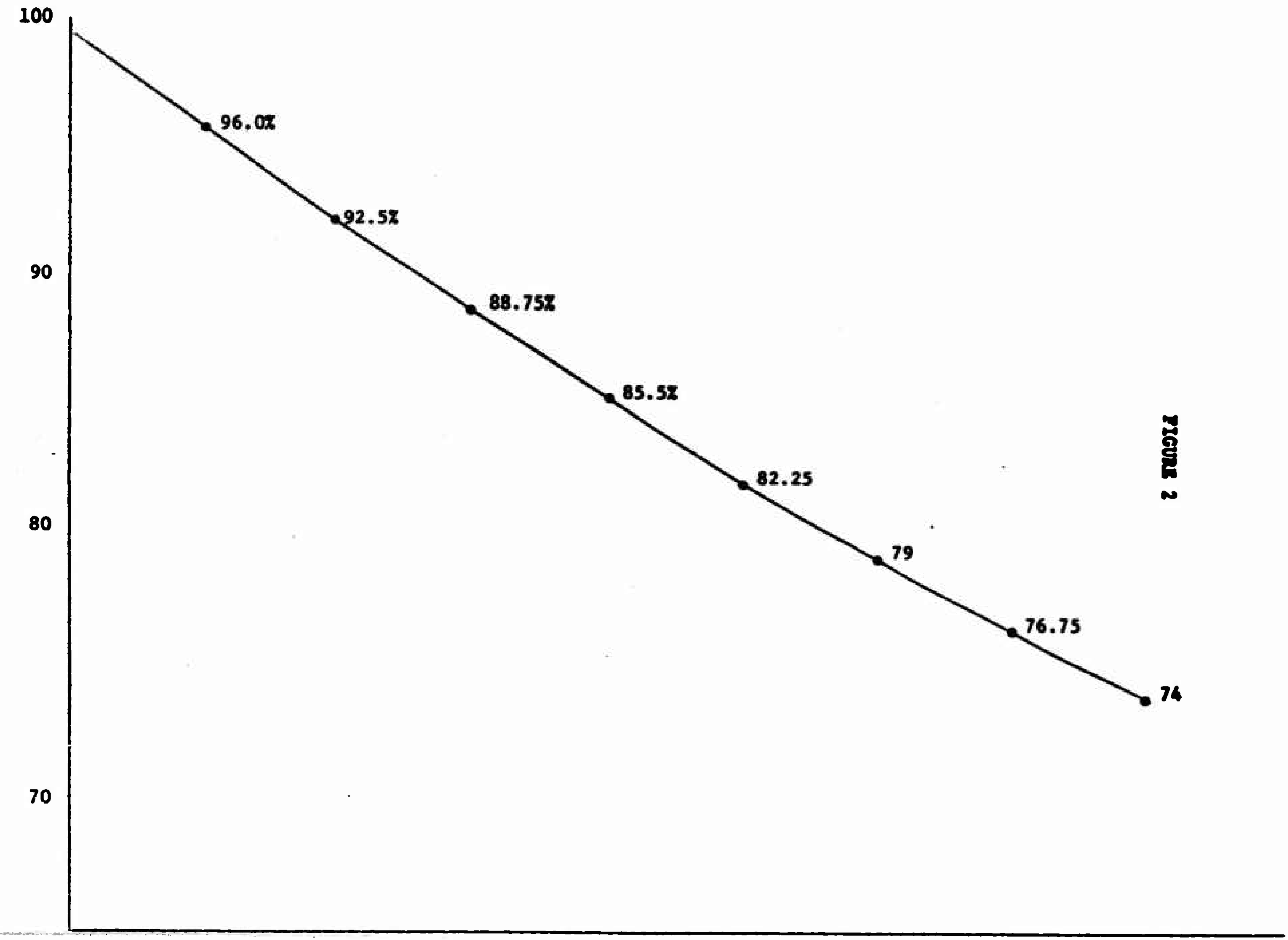


FIGURE 2

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